BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 4

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CONTENTS

I. REVIEW

Last time we:

(1) Reviewed the basics of Laurent series, which are doubly infinite series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

and converge on annuli.

(2) Discussed the three types of singularities that can occur for a holomorphic function defined on a punctured disc: removable singularities, poles, and essential singularities.

(3) Defined meromorphic functions on subsets of $\mathbb{C}$ as those with no worse than poles.

(4) Extended the definition of holomorphic and meromorphic functions to Riemann surfaces. A function $f : X \to \mathbb{C}$ is holomorphic (resp., meromorphic) iff $f \circ \varphi^{-1}$ is for every coordinate chart φ on X. We defined a morphism $h : X \to Y$ of Riemann surfaces similarly, by requiring that $\psi \circ h \circ \varphi^{-1}$ be holomorphic for every chart φ on X and ψ on Y.

II. MORPHISMS OF RIEMANN SURFACES (CONT.)

Let X be a Riemann surface, $P \in X$, and $f : X \to \mathbb{C}$ be a function. We defined f to be holomorphic at P if $f \circ \varphi^{-1} : U \to \mathbb{C}$ is holomorphic for every coordinate chart (U, φ) containing P. But as the next result shows, it actually suffices to check on just one chart containing P.

Lemma 1. *Let X be a Riemann surface, $P \in X$, and $f : X \to \mathbb{C}$ be a function such that there exists a coordiante chart (U, φ) containing P such that $f \circ \varphi^{-1} : U \to \mathbb{C}$ is holomorphic. Then f is holomorphic at P.*

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Proof. Given another chart (V, ψ) containing P, then

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})$$

wherever these compositions are defined. Since φ and ψ are both charts near P, then they are holomorphically compatible, so $\varphi \circ \psi^{-1}$ is holomorphic. Thus $f \circ \psi^{-1}$ is the composition of holomorphic maps, hence is holomorphic. □

Remark 1. The analogous fact is true of holomorphic maps $f : X_1 \to X_2$ of Riemann surfaces: we can replace the "for all" in the definition with a "there exists". More precisely, we can check that f is holomorphic by checking on specific open covers of X_1 and X_2.

II.1. **More automorphism groups, more complex analysis.** Last time we determined $\mathrm{Aut}(\mathbb{P}^1)$ and $\mathrm{Aut}(\mathbb{C})$.

Proposition 2. *The automorphism groups of $\mathbb{P}^1$ and $\mathbb{C}$ are:*

$$\begin{aligned}\mathrm{Aut}(\mathbb{P}^1) &= \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C}, ad-bc \neq 0 \right\} \\ &\cong \mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C}) \\ \mathrm{Aut}(\mathbb{C}) &= \{ z \mapsto az+b : a,b \in \mathbb{C} \} .\end{aligned}$$

This time we'll compute $\mathrm{Aut}(\mathfrak{H})$ and $\mathrm{Aut}(\mathfrak{D})$.

Proposition 3. *The automorphism groups of $\mathfrak{D}$ and $\mathfrak{H}$ are:*

$$\begin{aligned}\mathrm{Aut}(\mathfrak{D}) &= \left\{ z \mapsto \frac{\overline{a}z+\overline{b}}{bz+a} : a,b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\} \\ &= \left\{ z \mapsto e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z} : \alpha \in \mathbb{C}, |\alpha| < 1, \theta \in \mathbb{R} \right\} \\ &\cong \mathrm{PSU}_{1,1}(\mathbb{R})\,; \\ \mathrm{Aut}(\mathfrak{H}) &= \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R}, ad-bc = 1 \right\} \\ &\cong \mathrm{PSL}_2(\mathbb{R})\end{aligned}$$

To prove this, we'll need the following results from complex analysis.

Theorem 4 (Maximum modulus principle). *Let $U \subseteq \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be holomorphic. Suppose that there exists a point $z_0 \in U$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in U$. Then f is constant.*

Remark 5. The proof of this result uses the Cauchy integral formula to show that the sets

$$V_1 := \{z \in U : |f(z)| = |f(z_0)|\} \quad V_2 := \{z \in U : |f(z)| < |f(z_0)|\}$$

are both open. Since U is connected and $z_0 \in V_1$, then $V_2 = \varnothing$. Thus $|f|$ is constant, and by considering the real- and imaginary parts of f one can use the Cauchy-Riemann equations to show that this implies that f itself is constant.

Corollary 6. *Let $D \subseteq \mathbb{C}$ be a bounded domain and let $f : \overline{D} \to \mathbb{C}$ be a continuous function that is holomorphic on D. Then $|f(z)|$ obtains a maximum on some point of the boundary ∂D.*

Proof. Since $\overline{D}$ is compact and $|f|$ is continuous, then it attains a maximum. If the maximum occurs at some $z_0 \in D$, then f is constant on D, hence constant on $\overline{D}$ by continuity. Thus in this case $|f|$ attains a maximum, its only value, on every point of ∂D.

Otherwise $|f|$ attains a maximum on $\overline{D} \setminus D = \partial D$. $\square$

Lemma 2 (Schwarz)**.** *Suppose that f is holomorphic on the open unit disc $D = D(0,1)$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in D$. Furthermore, if $|f'(0)| = 1$ or $|f(w)| = |w|$ for some $w \in D$, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for all $z \in D$.*

Proof. Consider the function

$$g : D \to \mathbb{C}$$
$$z \mapsto \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0\,. \end{cases}$$

We claim that g is holomorphic on D. Note that g is holomorphic on the punctured disc $D^* := D^*(0,1)$ and since

$$g(0) = f'(0) = \lim_{z \to 0} \frac{f(z)}{z}$$

then g is also continuous at $z = 0$. By Riemann's removable singularity theorem, then g is holomorphic on all of D.

Given $z \in \mathfrak{D}$, fix some $r \in \mathbb{R}$ with $|z| < r < 1$. By the previous corollary, $|g|$ attains a maximum on ∂D, so

$$|g(z)| \leq \max_{|\zeta|=r} |g(\zeta)| = \max_{|zeta|=r} \frac{|f(\zeta)|}{|\zeta|} \leq \frac{1}{r}\,.$$

Since r was arbitrary, we can let $r \to 1$ which implies that $|g(z)| \leq \lim_{r \to 1} 1/r = 1$. Since $z \in D$ was arbitrary, then $|g(z)| \leq 1$ for all $z \in D$. Then

$$|f(z)| = |g(z)||z| \leq |z|$$

for all $z \in D$ and $|f'(0)| = |g(0)| \leq 1$.

Moreover, we must have $|g(z)| < 1$ for all $z \in D$, unless g is a constant function $g(z) = c$ for some $c \in \mathbb{C}$ with $|c| = 1$. This means that $|f'(0)| < 1$ and $|f(z)| < |z|$ for all $z \in D^*$, unless $f(z) = cz$ with $|c| = 1$. $\square$

With these results in hand, we can now prove Proposition 2.

Proof of Proposition 2. Suppose $f : \mathfrak{D} \to \mathfrak{D}$ is an automorphism and let $\lambda = f(0)$. We claim that $M(z) = \dfrac{z - \lambda}{1 - \overline{\lambda} z}$ preserves $\mathfrak{D}$. Given $z \in \mathfrak{D}$, then $|z| \leq 1$. Then

$$|M(z)|^2 = \frac{z - \lambda}{1 - \overline{\lambda} z} \overline{\frac{z - \lambda}{1 - \overline{\lambda} z}} = \frac{|z|^2 - \lambda \overline{z} - \overline{\lambda} z + |\lambda|^2}{1 - \lambda \overline{z} - \overline{\lambda} z + |\lambda|^2 |z|^2}\,.$$

Thus $|M(z)|^2 \le 1 \iff$

$$\frac{|z|^2 - \lambda\bar{z} - \overline{\lambda}z + |\lambda|^2}{1 - \lambda\bar{z} - \overline{\lambda}z + |\lambda|^2|z|^2} \le 1 \iff |z|^2 \cancel{-\lambda\bar{z} - \overline{\lambda}z} + |\lambda|^2 \le 1 \cancel{-\lambda\bar{z} - \overline{\lambda}z} + |\lambda|^2|z|^2$$
$$\iff |z|^2(1 - |\lambda|^2) = |z|^2 - |\lambda|^2|z|^2 \le 1 - |\lambda|^2$$

and this last inequality holds because $|z|^2 \le 1$. Thus $M(\mathfrak{D}) \subseteq \mathfrak{D}$ and $M(\lambda) = 0$.

Note that $M : \mathfrak{D} \to \mathfrak{D}$ is an automorphism. Since

$$\begin{pmatrix} 1 & -\lambda \\ -\overline{\lambda} & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ \overline{\lambda} & 1 \end{pmatrix} = \begin{pmatrix} 1 - |\lambda|^2 & 0 \\ 0 & 1 - |\lambda|^2 \end{pmatrix}$$

we see that

$$M^{-1}(z) = \frac{z + \lambda}{\overline{\lambda}z + 1}\,.$$

Letting $h = M \circ f$ then h is an automorphism, and thus so is h^{-1}. By Schwarz's Lemma applied to h, we have $|h(z)| \le |z|$ for all z, and applying it to h^{-1}, then $|h^{-1}(w)| \le |w|$ for all w. Writing $w = h(z)$, then

$$|z| = |h^{-1}(h(z))| = |h^{-1}(w)| \le |w| = |h(z)|$$

for all $z \in \mathfrak{D}$, so $|h(z)| = |z|$ for all $z \in \mathfrak{D}$. Applying the second part of Schwarz's Lemma, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $h(z) = cz$ for all z. Writing $c = e^{i\theta}$ for some $\theta \in \mathbb{R}$, then $(M \circ f)(z) = e^{i\theta}z$ for all z. Composing with M^{-1}, we find

$$f(z) = M^{-1} \circ M \circ f(z) = M^{-1}(e^{i\theta}z) = \frac{e^{i\theta}z + \lambda}{\overline{\lambda}e^{i\theta}z + 1} = e^{i\theta}\frac{z + e^{-i\theta}\lambda}{1 + e^{i\theta}\overline{\lambda}z}$$
$$= e^{i\theta}\frac{z - \alpha}{1 - \overline{\alpha}z}$$

where $\alpha = -e^{i\theta}\lambda$. (The other form for the matrices can be obtained from this one, too.)

As you will show in the homework, $\mathfrak{H}$ and $\mathfrak{D}$ are isomorphic via the map $T : z \mapsto \dfrac{z - i}{z + i}$. Thus given $f \in \operatorname{Aut}(\mathfrak{H})$, then $T \circ f \circ T^{-1} \in \operatorname{Aut}(\mathfrak{D})$. Thus f is a Möbius transformation. Note that T maps the real line to the unit circle: given $t \in \mathbb{R}$, then

$$f(t) = \frac{t - i}{t + i} = \frac{t - i}{\overline{t - i}}\,.$$

A complex number divided by its conjugate always has norm 1, so $T(\mathbb{R})$ is contained in the unit circle. These facts can be used to show that

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})\,.$$

□

Remark 7.

(1) We saw last time that $\operatorname{Aut}(\mathbb{P}^1) \cong \mathrm{PSL}_2(\mathbb{C})$. One can show that $\operatorname{Aut}(\mathbb{P}^1)$ acts triply transitively on $\mathbb{P}^1$: given $z_1, z_2, z_3, w_1, w_2, w_3 \in \mathbb{P}^1$, there exists $M \in \mathrm{PSL}_2(\mathbb{C})$ such that $M(x_j) = w_j$ for $j = 1, 2, 3$.

(2) We defined the torus $\mathbb{C}/\Lambda$ by quotienting $\mathbb{C}$ by the subgroup

$$\{z \mapsto z + \omega : \omega \in \Lambda\} \subseteq \mathrm{Aut}(\mathbb{C}).$$

Later we will consider quotients of $\mathfrak{H}$ and $\mathfrak{D}$ by subgroups of their automorphism groups.

II.2. **Order of vanishing and ramification.** In a neighborhood of a point $P \in X$, a morphism $F : X \to Y$ is an m-to-1 map. We can make this idea precise using the notion of multiplicity.

Definition 8. Let X be a Riemann surface, $P \in X$, and $f \in \mathcal{M}(X)$ be a meromorphic function. Let φ be a centered coordinate map at P, so $\varphi(P) = 0$. Then f can represented by the Laurent series $f \circ \varphi^{-1}(z) = \sum_n a_n z^n$. The order (of vanishing) of f at P, denoted by $\mathrm{ord}_P(f)$ is the smalles n such that $a_n \neq 0$:

$$\mathrm{ord}_P(f) := \min\{n \in \mathbb{Z} : a_n \neq 0\}.$$

If $\mathrm{ord}_P(f) n \geq 1$, then f has a zero of order n at P and if $\mathrm{ord}_P(f) = -n < 0$, then f has a pole of order n at P.

Remark 9. One can show that the order is independent of the choice of coordinate chart.

Lemma 3. *Let $f, g \in \mathcal{M}(X)$ be meromorphic functions on a Riemann surface X. Then*

(1) $\mathrm{ord}_P(fg) = \mathrm{ord}_P(f) + \mathrm{ord}_P(g)$;
(2) $\mathrm{ord}_P(1/f) = -\mathrm{ord}_P(f)$;
(3) $\mathrm{ord}_P(f+g) \geq \min\{\mathrm{ord}_P(f), \mathrm{ord}_P(g)\}$.

Remark 10. This shows that ord_P is a discrete valuation on $\mathcal{M}(X)$ for each point P.

Definition 11. Let $f : X \to Y$ be a morphism of Riemann surfaces, $P \in X$. Let ψ be a chart of Y centered at $f(P)$, so $\psi(f(P)) = 0$. The integer $e_P(f)$ or $m_P(f)$ given by

$$e_P(f) := \mathrm{ord}_P(\psi \circ f)$$

is the ramification index or multiplicity of f at P. Equivalently,

$$e_P(f) = 1 + \mathrm{ord}_P(\psi \circ f)'$$

whether ψ is a centered chart or not.

If $e_P(f) \geq 2$, then $P \in X$ is ramification point or branch point of f, with ramification index $e_P(f)$. A branch value is the image of a ramification point. Equivalently, we say that f is ramified above $Q \in Y$ if there is some $P \in f^{-1}(Q)$ with $e_P(f) \geq 2$ and f is ramified at $P \in X$ if $P \in X$ and $e_P(f) \geq 2$.

By choosing our charts judiciously, we can actually find a local representation of a morphism of the form $z \mapsto z^m$.

Proposition 12 (Local Normal Form)**.** *Let $F : X \to Y$ be a nonconstant morphism of Riemann surfaces. Fix $P \in X$ and let $m = e_P(F)$. Then for every chart $\psi : V \to \widehat{V}$ on Y centered at $F(P)$, there exists a chart $\varphi : U \to \widehat{U}$ on X centered at P such that*

$$(\psi \circ F \circ \varphi^{-1})(z) = z^m.$$

Proof. Fix a chart ψ on Y centered at $F(P)$ (i.e., $\psi(F(P)) = 0$), and choose any chart $\theta : W \to \widehat{W}$ centered at P. Then the Taylor series for the function $T(w) := (\psi \circ F \circ \theta^{-1})(w)$ is of the form

$$T(w) = \sum_{j=m}^{\infty} c_j w^j$$

where $c_m \neq 0$ and $m = m_P(F)$. (Since we picked a centered chart, we have $T(0) = 0$.) Factoring out w^m, we have $T(w) = w^m S(w)$ where S is a holomorphic function at $w = 0$ and $S(0) \neq 0$. Thus we can define a branch of the m^{th} root function near $S(0)$, so there exists a holomorphic function R defined in a neighborhood of 0 such that $R(w)^m = S(w)$. Let $\eta(w) = wR(w)$, so

$$T(w) = w^m S(W) = (wR(w))^m = (\eta(w))^m .$$

Then

$$\eta'(w) = wR'(w) + R(w)$$

so $\eta'(0) = R(0) = \sqrt[m]{S(0)} \neq 0$, so near 0 η is invertible by the Implicit Function Theorem. Then $\varphi := \eta \circ \theta$ is also a chart on X defined near P, and since

$$\eta(\psi(P)) = \eta(0) = 0 \cdot R(0) = 0$$

it is also centered at P. Thinking of $z = \eta(w)$ as our new coordinate near P, then we have

$$(\psi \circ F \circ \varphi^{-1})(z) = (\psi \circ F \circ \theta^{-1} \circ \eta^{-1})(z) = T(\eta^{-1}(z)) = (\eta(\eta^{-1}(z)))^m = z^m .$$

□

Lemma 4. *Let $X : f(x,y) = 0$ be a smooth affine plane curve. Consider the projection $\pi : X \to \mathbb{C}, (x,y) \mapsto x$. Then π is ramified at $P = (x_0, y_0) \in X$ iff $f_y(P) = 0$.*

Proof. Suppose first that $f_y(P) \neq 0$. Then π is a chart on X near P, so π has multiplicity 1 at P. Conversely, suppose that $f_y(P) = 0$. Then $\rho : (x,y) \mapsto y$ is a chart on X near P. By the Implicit Function Theorem, then there exists a holomorphic function $g(y)$ such that X is locally the graph of g, so $f(g(y), y) = 0$ for all y in the domain of g. Implicitly differentiating with respect to y, we have

$$f_x(g(y), y) g'(y) + f_y(g(y), y) = 0$$

for all y, so in particular

$$0 = f_x(g(y_0), y_0) g'(y_0) + f_y(g(y_0), y_0) = f_x(P) g'(y_0) + f_y(P) = f_x(P) g'(y_0) .$$

Since X is smooth and $f_y(P) = 0$, then $f_x(P) \neq 0$, so we must have $g'(y_0) = 0$. □

Example 13. Let $E : Y^2 Z = X^3 - Z^3$ and consider the map $\pi : E \to \mathbb{P}^1, [X : Y : Z] \mapsto [X : Z]$. On the affine chart U_2 where $Z \neq 0$, E is given by the equation $y^2 = x^3 - 1$ where $x = X/Z$ and $y = Y/Z$. Denoting the homogeneous coordinates of $\mathbb{P}^1$ by S, T, then π carries U_2 to the open subset V_1 of $\mathbb{P}^1$ where $T \neq 0$. On V_1 we have the affine coordinate S/T, so the local expression of π as a map $U_2 \to V_1$ is simply $(x, y) \mapsto x$. Letting $h = y^2 - (x^3 - 1)$, by the above lemma, π is unramified at all points where $h_y = 2y \neq 0$. Thus it remains to consider the points where $y = 0$, consisting of $(\zeta^j, 0)$ for $j = 0, 1, 2$, where ζ is a primitive third root of unity.

At such a point the projection $(x, y) \mapsto y$ is a coordinate chart, so there exists a holomorphic function $g(w)$ such that

$$0 = h(g(w), w) = w^2 - (g(w)^3 - 1)$$

and $g(0) = \zeta^j$. Write $g(w) = \sum_{n \geq 0} a_n w^n$, so $a_0 = g(0) = \zeta^j$. Differentiating the above, we find

$$0 = h_x(g(w), w) g'(w) + h_y(g(w), w) = 3g(w)^2 g'(w) + 2w$$
$$\implies g'(w) = \frac{-2w}{3g(w)} = -\frac{2}{3} \frac{w}{g(w)} .$$

Thus

$$a_1 = g'(0) = -\frac{2}{3} \frac{0}{g(0)} = -\frac{2}{3} \frac{0}{\zeta^j} = 0$$

so $a_1 = 0$, as expected. Differentiating again, we find

$$g''(w) = -\frac{2}{3} \frac{g(w) - w g'(w)}{g(w)^2} .$$

Then

$$2a_2 = g''(0) = -\frac{2}{3} \frac{g(0) - 0 \cdot g'(0)}{g(0)^2} = -\frac{2}{3} \frac{a_0}{{a_0}^2} = -\frac{2}{3} \frac{\zeta^j}{\zeta^{2j}} = -\frac{2}{3} \zeta^{-j} .$$

Thus $m = 2$ is the smallest $n \geq 1$ such that $a_n \neq 0$, so π has ramification index $e_P(\pi) = 2$ for $P = (\zeta^j, 0)$. (There's one other point we need to check; what is it?)